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## A Comparison of N-Path and Digital Filters for Band Pass Applications

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A COMPARISON OF N-PATH  
AND DIGITAL FILTERS FOR  
BAND PASS APPLICATIONS

BY  
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RESEARCH REPORT

Submitted in partial fulfillment of the requirements for the degree of  
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Orlando, Florida  
1974

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# A COMPARISON OF N-PATH AND DIGITAL FILTERS FOR BAND PASS APPLICATIONS

MR. STEPHEN COOPERMAN

## ABSTRACT

The classic problem of achieving high  $Q$ , band pass, inductorless filters has been traditionally attacked by applying active RC networks. This approach suffers from the faults of poor economy, lack of stability of performance with time and temperature, and high network performance sensitivity to component changes.

The problem is reviewed from the aspect of applying techniques other than operational amplifier RC filters. The techniques presented are those of N-Path (switching) and Digital Filters.

The underlying theory of both methods is explored and a design example for each of the techniques is presented. These designs are then compared on the basis of design complexity, parts cost (count) and ease of fabrication. On the basis of the comparison of designs, it is shown which technique is more economical for the problem under discussion.



### Acknowledgement

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## INTRODUCTION

The application of conventional active filter theory to the class of problems in the relatively low frequency domain, often gives rise to difficulties in achieving sharp cut off and other important criteria.

The techniques that have been applied to overcome the above difficulties include Crystal Filters, Digital Filters, and a class of filters referred to in the literature as Commutating, Switching or N-Path Filters. It also appears, from the abundance of articles in the trade journals, that Digital Filters are becoming a panacea for this type of application.

This work is undertaken to demonstrate that no one design approach is a sure remedy for a particular engineering environment, but rather that there are other tractable alternatives to Digital Filters for this low frequency, high Q problem. The alternative is N-Path Filters.

At this point some definitions are in order so that the classes of filters to be discussed may be more easily understood.

Digital Filter—A class of filters in which either a computer program or special purpose digital hardware provide an approximation to a continuous filter characteristic. In the hardware approach, sampling and quantization provide a sequence of numbers representing the input waveform. The hardware operates upon these sequences to provide the desired approximation. Then the output sequences are usually converted back into analog form for further use.



N-Path Filter—A class of filters in which  $N$  identical parallel signal paths and input and output samplers provide, under certain conditions, a transfer function which may be tailored into a desired filter characteristic. The  $N$  identical signal paths may contain both active and passive components. The filter is usually followed by a low pass network.

There is a need to show that there is an alternative to Digital Filters in order that design and hardware costs may be kept in order, and that competitive designs, concurrent with modern integrated circuitry, be provided. This is the subject of this work.

This report is divided as follows:

1. Theory of N-Path Filters.
2. Theory of Digital Filters.
3. N-Path Filter Design Example.
4. Digital Filter Design Example.
5. Design Comparison and Analysis.
6. Conclusions.

The theory will be presented as an overview in an attempt to unify the existing literature and to provide a basis for future efforts. In this sense, it will not be all encompassing, but rather, will give a means of comparison of the two approaches to the problem. In this same light, the design examples are chosen for ease of comparison of their salient features and not for complexity. It is the objective of this effort to allow alternatives to be evaluated, rather than presenting complex designs, in which important points for comparison would be lost in detail.



## I. N-PATH FILTER THEORY

The purpose of this section is to present an overview of the existing theory of the N-Path Filter. This theory will be presented in such a manner that the material may be applied to later design of such filters in a more unified fashion.

N-Path Filters are time varying networks consisting of  $N$  identical signal paths. Each path within the overall network contains an input modulator, a time invariant active or passive network, and an output modulator.

A single path within the filter may be represented as shown in figure 1.

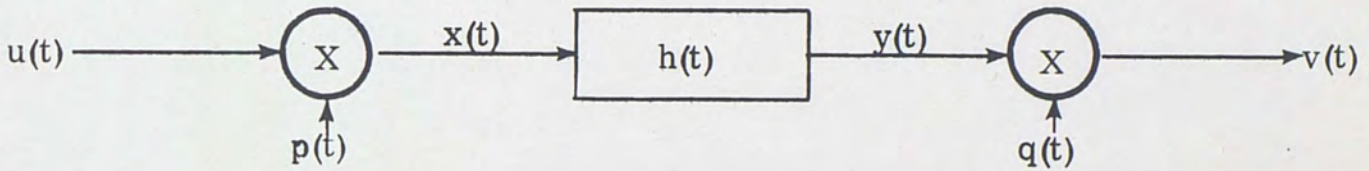


Figure 1. Single Path, N-Path Structure

The theory of these single paths, and the overall network, may be approached from either the time or frequency domain. Franks [4] uses the time domain approach in his later work, but in earlier efforts [3], used the frequency domain in arriving at a description of the network. It is quite instructive to arrive at the network description from both domains and compare results. To this end, we will first examine this single path of figure 1 from the time domain and then the frequency domain.



$P(t)$  is a periodic signal, which may be represented by its Fourier Series;

$$p(t) = \sum_{p=-\infty}^{\infty} P_p \exp [(j2\pi pt)/N\tau], \quad (1)$$

where  $p(t)$  is periodic in  $N\tau$  and  $P_p$  are the Fourier coefficients which are wave-form dependent.  $N$  is the number of paths and  $h(t)$  is the impulse response of the time invariant path network.

Similarly;

$$q(t) = \sum_{\ell=-\infty}^{\infty} Q_{\ell} \exp [j(2\pi \ell t)/N\tau]. \quad (2)$$

$v(t)$  will equal the convolution of the impulse response of the path, and the input,  $u(t)$ . Define this impulse response as  $W(t, t_1)$ . Then;

$$v(t) = \int_{-\infty}^{\infty} W(t, t_1) u(t_1) dt_1. \quad (3)$$

If it is assumed that the time invariant path network is realizable, then it must be non anticipatory. If the impulse is applied at time  $\tau$ , the path network response must be  $h(t-\tau)$  to satisfy the condition of being non anticipatory. The impulse response for an input at time  $\tau$  will be

$$W(t, t_1) = W(t, \tau)^1 = p(\tau) h(t-\tau) q(t). \quad (4)$$

This follows, if it realized that the input modulator acts upon the unit impulse applied at time  $\tau$ , as a scale factor  $p(\tau)$ ; the network provides its impulse response, and the product of the two is acted upon by the output modulator at time  $t$ .

---

<sup>1</sup>Again, to satisfy realizability constraints, the network must not anticipate future input values;  $t - \tau \geq 0$



Consider an input of the form,  $\exp (St)$ , where  $S$  is the complex variable  $\sigma + j\omega$ . Then the convolution will take the form;

$$v(t) = \int_{-\infty}^t p(\tau) h(t - \tau) q(t) \exp (S\tau) d\tau, \quad (5)$$

where the upper limit has been replaced by  $t$  because of the assumed realizability of the network.

Change variables as follows;

$$\text{let } A = t - \tau; \quad \tau = t - A$$

$$\text{Upper limit} = t \Rightarrow A = 0$$

$$\text{Lower limit} = -\infty \Rightarrow A = \infty$$

$dA = -d\tau$ ; therefore, since linear network, reverse limits of integration.

$$v(t) = \sum_{\ell} \sum_P Q_{\ell} P_P \exp \left[ j \frac{2\pi t}{N\tau} (\ell + P) \right] \exp (St) \int_0^{\infty} h(A) \exp \left( -j \frac{2\pi PA}{N\tau} - SA \right) dA. \quad (6)$$

It is known that the integral represents

$$H \left( S + j \frac{2\pi P}{N\tau} \right),$$

where  $H(S)$  is the La Place transform of  $h(t)$ . Then;

$$v(t) = \left\{ \sum_{\ell} \sum_P Q_{\ell} P_P \exp \left[ j \frac{2\pi t}{N\tau} (\ell + P) \right] H \left( S + j \frac{2\pi P}{N\tau} \right) \right\} \exp (St). \quad (7)$$

For a time invariant network, the response would be proportional only to the input,  $\exp (St)$ . But, because the network is time varying, there are additional components in the response which are images of the input. If the complex variable  $S$  is confined to the  $j\omega$  axis, then these images are at  $(f = P/N\tau)$ . Their magnitude depends upon the Fourier coefficients  $P_P$ ,  $Q_{\ell}$  and the magnitude of  $H(j\omega)$  at  $\omega = 2\pi P/N\tau$ .



The basic operating principle of N-Path structures is that in combining the multipath responses, the images can be phase cancelled over any band by choosing N large enough. The result is that within this band, the characteristic is a time invariant filter.

Consider again the same single path depicted in figure 1. In the frequency domain,  $V(S)$  will be the convolution of  $Y(S)$  and  $Q(S)$ ;

$$V(S) = Y(S) * Q(S), \quad (8)$$

$$Q(S) = \int_0^{\infty} q(t) \exp(-St) dt, \quad (9)$$

$$Q(S) = \int_0^{\infty} \sum_{\ell} Q_{\ell} \exp\left(j \frac{2\pi \ell t}{N\tau}\right) \exp(-St) dt, \quad (10)$$

$$Q(S) = \sum_{\ell} \frac{Q_{\ell}}{S - j \frac{2\pi \ell}{N\tau}} \quad (11)$$

Then;

$$V(S) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} Y(S - \xi) Q(\xi) d\xi. \quad (12)$$

Now let the closed contour encircle only the poles of  $Q(\xi)$ :  $\xi = j 2\pi \ell / N\tau$ ; the residues there are;

$$\text{Residues} = \sum_{\ell} Q_{\ell} Y\left(S - j \frac{2\pi \ell}{N\tau}\right) \quad (13)$$

The value of the integral is,

$$V(S) = 2\pi j \times \frac{1}{2\pi j} \times \sum_{\ell} Q_{\ell} Y\left(S - j \frac{2\pi \ell}{N\tau}\right), \quad (14)$$



$$V(S) = \sum_{\ell} Q_{\ell} Y\left(S - j \frac{2\pi\ell}{N\tau}\right). \quad (15)$$

But  $Y(S)$  is equal to  $X(S) H(S)$ . Therefore;

$$V(S) = \sum_{\ell} Q_{\ell} X\left(S - j \frac{2\pi\ell}{N\tau}\right) H\left(S - j \frac{2\pi\ell}{N\tau}\right). \quad (16)$$

Similarly, using convolution again for  $X(S)$  in terms of  $U(S)$  and  $P(S)$ ;

$$V(S) = \sum_{\ell} \sum_P P_P Q_{\ell} U\left[S - j \frac{2\pi(P+\ell)}{N\tau}\right] H\left(S - j \frac{2\pi\ell}{N\tau}\right). \quad (17)$$

Again, it is seen that images of the input will occur. These may again be phase cancelled by suitable choice of  $N$ .

It is interesting to compare the two approaches at this time. Notice, that with the time domain approach, a particular input was chosen in order to illustrate the imaging that results in the path. In the frequency domain avenue of arriving at the path characteristic, no particular input needed to be chosen. This is significant, and again illuminates the utility of the frequency domain in the solution of engineering problems in general. This method is to be preferred when dealing with this class of problem.

The effect of the combined structure, all  $N$  paths, on the overall network response must be considered next. The entire  $N$ -Path structure is shown in figure 2.



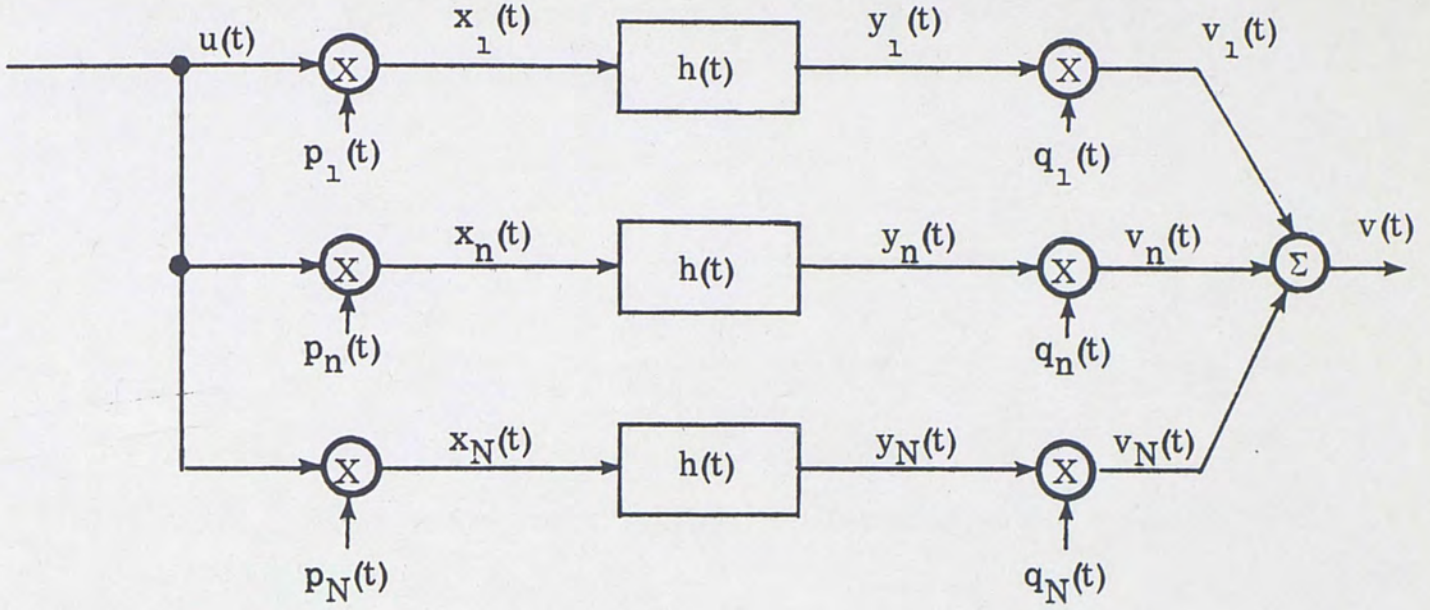


Figure 2. N-Path Structure

In analyzing the combined network, the frequency domain will be employed.

$P_n(t)$  is periodic in  $N\tau$  and may be represented by its Fourier Series;

$$P_n(t) = \sum_p P_p \exp \left( j \frac{2\pi p t}{N\tau} \right) \exp \left[ -j \frac{2\pi p (n-1)\tau}{N\tau} \right]. \quad (18)$$

Similarly,  $Q_n(t)$ , also periodic in  $N\tau$ , may be represented by its Fourier Series;

$$Q_n(t) = \sum_\ell Q_\ell \exp \left( j \frac{2\pi \ell t}{N\tau} \right) \exp \left[ -j \frac{2\pi \ell (n-1)\tau}{N\tau} \right]. \quad (19)$$

Then  $V_n(S)$  is the convolution of  $Y_n(S)$  with  $Q_n(S)$ .  $Q_n(S)$  is given by;

$$Q_n(S) = \left\{ \sum_\ell Q_\ell \exp \left[ -j \frac{2\pi \ell (n-1)\tau}{N\tau} \right] \right\} \frac{1}{S - j \frac{2\pi \ell}{N\tau}}. \quad (20)$$



Performing the convolution and using a closed contour about the poles of  $Q_n(S)$  yields;

$$V_n(S) = Y_n \left( S - j \frac{2\pi\ell}{N\tau} \right) \sum_{\ell} Q_{\ell} \exp \left[ -j \frac{2\pi\ell(n-1)\tau}{N\tau} \right]. \quad (21)$$

Again,  $Y_n(S)$  equals the product of  $X_n(S)$  and  $H(S)$ . Also, as noted previously,  $X_n(S)$  is the result of the convolution of  $U(S)$  with  $P(S)$ . Performing the required operations gives  $V_n(S)$  in the following form:

$$V_n(S) = \sum_{\ell} \sum_p P_p Q_{\ell} \exp \left[ -j \frac{2\pi(n-1)(P+\ell)\tau}{N\tau} \right] U \left[ S - j \frac{2\pi(P+\ell)}{N\tau} \right] H \left( S - j \frac{2\pi\ell}{N\tau} \right). \quad (22)$$

The overall structure response may be found by summing over the  $N$ -Paths;

$$V(S) = \sum_{n=1}^N \sum_{\ell} \sum_p P_p Q_{\ell} \exp \left[ -j \frac{2\pi(n-1)(P+\ell)\tau}{N\tau} \right] U \left[ S - j \frac{2\pi(P+\ell)}{N\tau} \right] H \left( S - j \frac{2\pi\ell}{N\tau} \right). \quad (23)$$

In this summation over  $n$ , the exponential portion yields a geometric series which sums to  $N$  for integer values of  $(P+\ell)$  multiplied by  $N$ , and sums to zero otherwise. Letting  $P+\ell$  equal  $KN$  and substituting for the index  $P$ , finally yields;

$$V(S) = N \sum_K \sum_{\ell} P_{KN-\ell} Q_{\ell} H \left( S - j \frac{2\pi\ell}{N\tau} \right) U \left( S - j \frac{2\pi KN}{N\tau} \right). \quad (24)$$

Consider the effect if the complex frequency variable  $S$ , is confined to the imaginary axis. Further, let the input signal be band limited to  $N/2T$ . It may be seen that the output images, for  $K \neq 0$ , will occur at  $\omega > 2\pi N/2T$ . Therefore, if the structure is preceded and followed by low pass filters with cutoff at or



below  $N/2T$ , the entire structure follows the  $K = 0$  path, and appears time invariant. This effect may also be produced by having the modulator signals band limited to  $N/2T$ . If this is done;

$$P_{\ell} = Q_{\ell} = 0 \text{ for all } \ell \text{ such that } |\ell| \geq N/2,$$

then,

$$P_{KN-\ell} Q_{\ell} = 0 \text{ for } K \neq 0 \text{ and the network follows the } K = 0 \text{ or time invariant path.}$$

If either of the above band limiting restrictions hold, then the only path of interest is the  $K = 0$  path. If we let  $K = 0$ , then  $V(j\omega)$  for the structure becomes;

$$V(j\omega) = N \sum_{\ell} P_{-\ell} Q_{\ell} H\left(j\omega - j\frac{2\pi\ell}{N\tau}\right) U(j\omega). \quad (25)$$

Thus, the transfer function consists of frequency translated versions of the path networks, scaled by  $N$ , and the modulation waveform dependent Fourier coefficients. If the networks are low-pass filters with cutoff at  $1/2 T$ , then the  $N$ -Path transfer characteristic is a multiple passband shape with equally spaced sidebands (passbands). This is how one might realize a bandpass filter structure.

At this juncture the equations pertaining to the  $N$ -Path structure may be suitably expanded upon by the inclusion of specific modulating functions and path networks. However, the purpose of this development is to provide a unified overview of the  $N$ -Path theory so that intelligent alternatives in bandpass filter design may be realized. This purpose is best served by a discussion of the results obtained so far and postponing actual function selection until the design example section.

It has been shown (25) that the overall  $N$ -Path response may be made to follow a single path, called the zeroth path. The characteristic for the network



resembles that of a time invariant filter. To achieve this, the N-Path structure requires a preceding and following low pass of filter or requires that the modulating functions be band limited.

The band limiting restriction is in reality not severe. The domain of usefulness of this class of structures is from about 5 Hz to 15 MHz [1]. In this band, the required low-pass filters may be simple R-C networks. These filter types are not normally prone to sensitivity problems and therefore would not add to the design burden.

It may be noticed from (25) that the structure may easily track a changing center frequency by varying the modulation rate. This feature is not simple to attain in other types of filter structures.

Band limiting for the modulation functions may be easily accomplished by no more elegant a scheme than using sinusoidal waveforms.

The overall network characteristic (25) may look unwieldy because of the lack of a closed form. This is not a problem, however, as a closed form may be easily obtained once a particular path network is chosen [4].

The widest use of these structures is with low-pass path networks realizing an overall bandpass characteristic. It is this application that will be explored in the design example section.

In summary, the salient points of N-Path structures are:

1. Require band limiting.
2. Achieve time invariance because of band limiting.
3. Overall characteristic depends upon modulating waveform for scaling.
4. Structure characteristic is a frequency translated, amplitude scaled version dependent upon the zeroth ( $K = 0$ ) path.



## II. DIGITAL FILTER THEORY

This section is concerned with the underlying theory of Digital Filters. This development is based upon the approximation of a continuous filter characteristic by a quantized one and relies heavily upon the ideal or impulse sampler. It is appropriate then, to review the impulse sampling process, and its mathematical concepts.

Consider a signal  $g(t)$ , applied to a product modulator as shown in figure 3. The modulating function is  $m(t)$ ;

$$m(t) = \sum_{n=0}^{\infty} \delta(t - nT), \quad (26)$$

an impulse train.

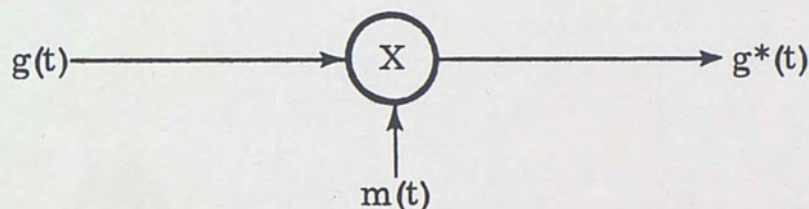


Figure 3. Ideal Sampling

The output signal,  $g^*(t)$ , is the product of  $g(t)$  and the modulating function. In the time domain, this output signal is just;

$$g^*(t) = g(t) \sum_{n=0}^{\infty} \delta(t - nT) = \sum_n g(nT) \delta(t - nT). \quad (27)$$



Thus, the output is a weighted impulse train, the weight being the value of the sampled function at the sampling instant. In this output, nothing is known about  $g(t)$  at times other than the sampling instants.

The process of ideal sampling also changes the frequency spectrum of the sampled signal. This may be shown by looking at this operation in the frequency domain. Consider the La Place transformation of the sampled signal.

$$G^*(S) = \mathcal{L}[g^*(t)] = \mathcal{L}[g(nT) \delta(t - nT)]. \quad (28)$$

But we realize that this is a weighted impulse train and therefore,

$$G^*(S) = g(nT) \sum_n \exp(-SnT). \quad (29)$$

This summation has a closed form which is found easily by long division as,

$$\sum_n \exp(-SnT) = \frac{1}{1 - \exp(-ST)}. \quad (30)$$

Then we may express the above transformation as,

$$G^*(S) = \frac{g(nT)}{1 - \exp(-ST)}. \quad (31)$$

The signal  $g(t)$  now has added to its complex frequency domain characteristics an infinite number of simple poles which are periodic in nature. These poles occur at,

$$S = \pm j \frac{2\pi n}{T}, \quad (32)$$

and are periodic at  $2\pi n/T$  or  $n\omega_s$ , where  $\omega_s$  is the angular sampling frequency.

Consider again obtaining the La Place transform  $G^*(S)$ , but this time approaching it from the viewpoint of complex convolution. We assume that  $g(t)$



is a function which has only left half plane poles for convenience. The complex convolution is,

$$G^*(S) = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} G(\omega) M(S - \omega) d\omega. \quad (33)$$

Now we close the contour to the right encircling the infinite number of poles of  $M(S - \omega)$ . We evaluate the integral as minus  $2\pi j$  times the sum of the residues of the integrand, at the poles of  $M(S - \omega)$ . The poles in this case are,

$$\omega_n = S + j \frac{2\pi n}{T}. \quad (34)$$

The residue at each of these simple poles is found by,

$$\text{Residue}_n = \frac{G(\omega_n)}{d/d\omega [1 - \exp(-ST) \exp(\omega T)]_{\omega = \omega_n}}. \quad (35)$$

This residue is,

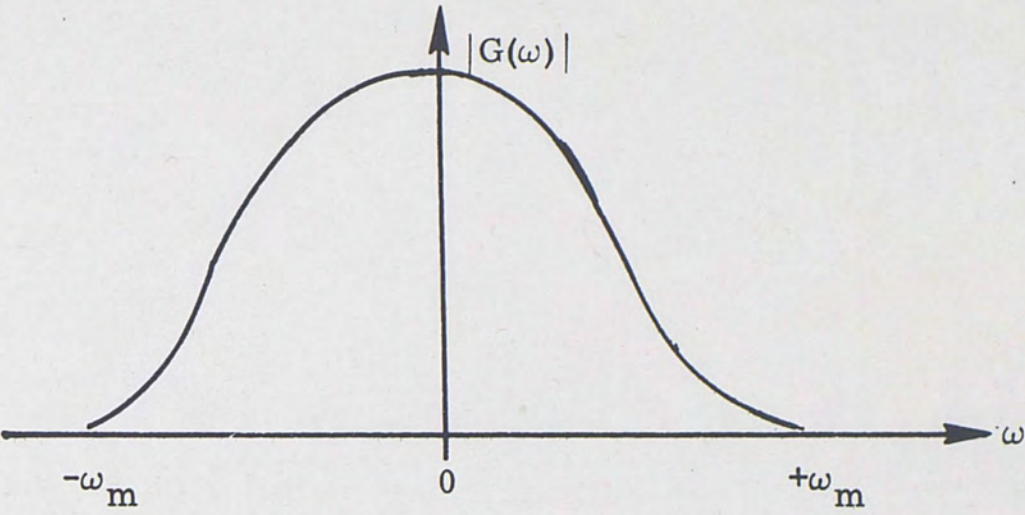
$$\text{Residue}_n = -\frac{1}{T} G\left(S + j \frac{2\pi n}{T}\right). \quad (36)$$

The summation of these residues leads to the form,

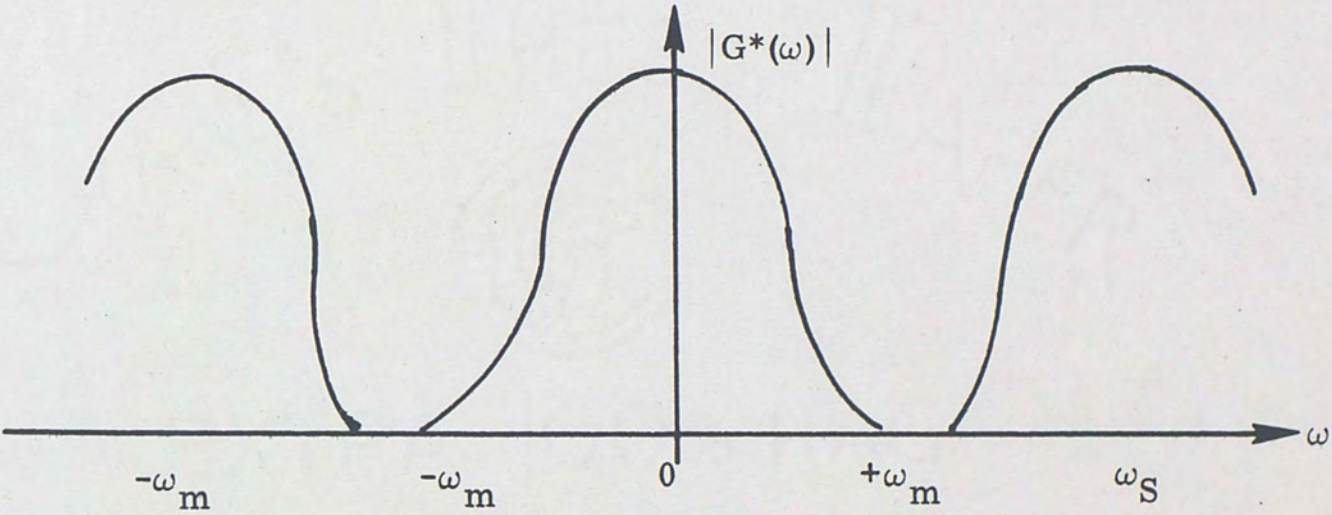
$$G^*(S) = \frac{1}{T} \sum_{n=-\infty}^{\infty} G\left(S + j \frac{2\pi n}{T}\right). \quad (37)$$

It may be seen then that sampled spectrum is periodic in  $\omega_s$  and gives rise to frequencies which were not present in the continuous signal. This phenomenon is called aliasing. It may be further noted that there is a critical sampling frequency below which the frequency domain characteristics are distorted by sampling. This is shown in figure 4, where it is assumed that the original signal is bandlimited to  $\omega_m$ . From that figure it is seen that in order to preserve the shape of the frequency characteristic, sampling must be done at least at a rate of  $2\omega_m$  [6].

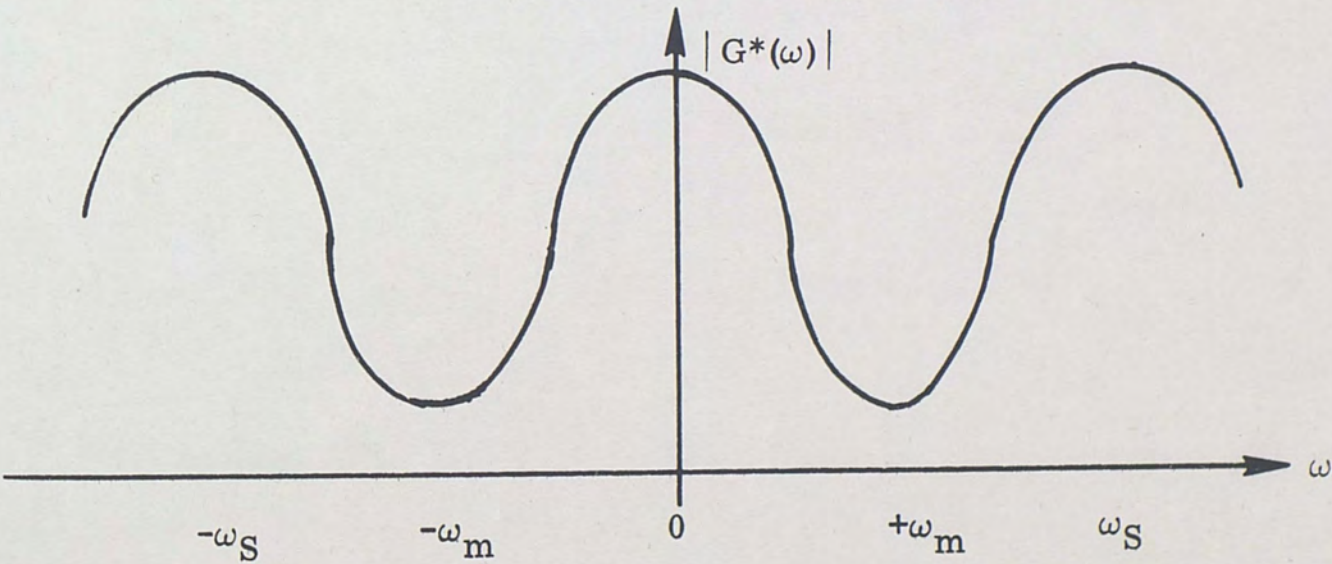




a. Original Spectrum



b. Sampled Spectrum  $\omega_S > 2\omega_m$



c. Sampled Spectrum  $\omega_S < 2\omega_m$

Figure 4. Aliasing and Effect of Sample Rate



Now that the process of ideal or impulse sampling has been reviewed, we may return to the Digital Filter problem. In this area one may consider that a continuous filter characteristic is to be approximated by the quantizing process of impulse sampling. Consider then, figure 5. We apply an impulse to the network and wish to obtain its quantized impulse response.  $P(t)$  is  $T_m(t)$  in order to preserve power after sampling [5].

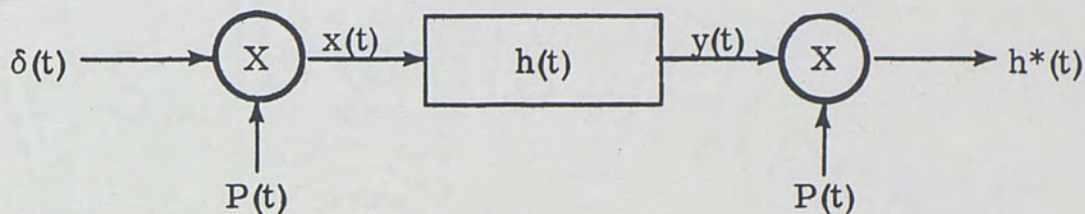


Figure 5. Basic Quantizing Structure

$x(t)$ , in the case of an impulse input is also an impulse as may be seen by applying complex convolution to the La Place transform of the product of  $\delta(t)$  with  $p(t)$ . Using the previous results of ideal sampling,

$$H^*(S) = T \sum_{n=0}^{\infty} h(nT) \exp(-SnT). \quad (38)$$

At this point it is convenient to introduce the Z transformation. If we define

$$Z \equiv \exp(ST), \quad (39)$$

then the sampled impulse response will take the form,

$$H^*(Z) = T \sum_{n=0}^{\infty} h(nT) Z^{-n}. \quad (40)$$

The concept of the Z transformation is a powerful one, taking the form of a mapping between points or regions in the complex frequency domain (S-plane) and regions in the sampled domain (Z plane). This mapping resolves the infinite number of poles introduced by sampling into a finite number of poles in the







approximation is truncated, the model possess only finite memory and is called non-recursive [5]. This name is applied to the truncated model because the impulse response is of finite duration and the output is not recursively related to past values of the output, but only to the last  $N$  values of the input.

A finite memory recursive filter may be developed if we again resort to complex convolution. In figure 5, let the La Place transform of  $h^*(t)$  be found,  $H^*(S)$ .

$$H^*(S) = \mathcal{L}[h^*(t)] = \frac{1}{2\pi j} \int_{-\infty}^{\infty} H(\omega) P(S - \omega) d\omega. \quad (41)$$

We assume that  $H(S)$  has only left half plane poles in the complex frequency domain ( $S$ -plane), and therefore it has these same poles in the dummy plane ( $\omega$ -plane). The restriction of left half plane singularities is applied because we are trying to quantize an impulse response of a characteristic that is realizable in continuous network theory. Hence, all the restrictions must apply. The integral will be evaluated by closing the contour to the left and evaluating residues at the poles of  $H(\omega)$ . It is assumed that there is a finite number of these;  $NP$  in number. The residue at each of these poles is,

$$\text{Residue}_n = \frac{TH'(\omega_n)}{1 - \exp(-ST) \exp(\omega_n T)}. \quad (42)$$

In (42)  $\omega_n$  is a pole of  $H(\omega)$  and  $H'(\omega_n)$  is the value of  $H(\omega)$  at  $\omega_n$  with the singularity removed. The residues are now summed over the  $NP$  poles of  $H(\omega)$ , and the factor  $2\pi j$  times this sum is used.  $H^*(S)$  is then given by,

$$H^*(S) = T \sum_{n=1}^{NP} \frac{H'(\omega_n)}{1 - \exp(-ST) \exp(\omega_n T)}. \quad (43)$$



If the Z transformation is employed,  $Z \equiv \exp (ST)$ , and if  $H'(\omega_n)$  is defined as  $A_n$ , and further, if  $-\exp (\omega_n T)$  is defined as  $B_n$ , the resultant function in the Z plane is,

$$H(Z) = T \sum_{n=1}^{NP} \frac{A_n}{1 + B_n Z^{-1}}. \quad (44)$$

This filter is called recursive because it is the closed form of the infinite summation and hence exhibits infinite memory of the input. It represents the quantized filter characteristic, the pulse (unit) response, and the transformed (Z) difference equation of the quantizing network. It is important to realize that if  $H(\omega)$  is a second order function with complex conjugate roots, then  $A_n$  will be in the form,

$$A_i Z^{-1} + A_j. \quad (45)$$

In order to visualize that a filter characteristic has been realized, consider the quantization of a simple low pass response. In the complex frequency domain,  $H(S)$  is,

$$H(S) = \frac{1}{1 + S\omega_n}. \quad (46)$$

The Z transform of this characteristic is then given by

$$H(Z) = \frac{T/\omega_n}{1 - Z^{-1} \exp (-T\omega_n)}. \quad (47)$$



Equation (44) may be modeled as a delay network. This is shown in figure 7.

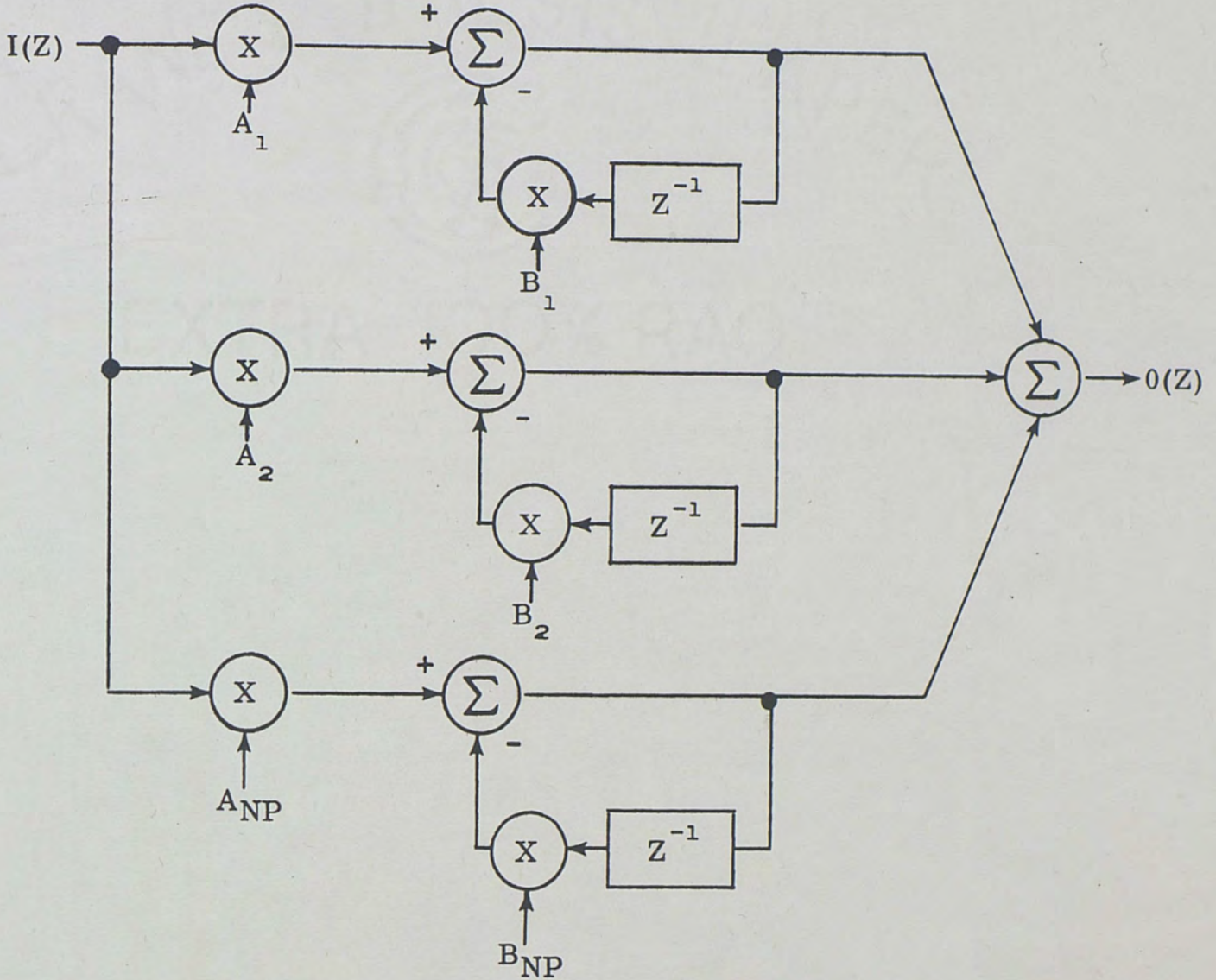


Figure 7. Basic Digital Filter Model

Let the complex frequency variable  $S$  be confined to the imaginary axis. Then  $Z \equiv \exp(ST)$  will equal  $\exp(j\omega T)$ . Then, as a function of frequency,

$$H[\exp(j\omega T)] = \frac{T/\omega_n}{1 - \exp(-j\omega T) \exp(-T/\omega_n)} \quad (48)$$

The magnitude function is then found to be,

$$|H[\exp(j\omega T)]| = \frac{1}{\omega_n/T [1 - 2 \exp(-T/\omega_n) \cos \omega T + \exp(-2T/\omega_n)]^{1/2}} \quad (49)$$



From (49) it may be seen that this function exhibits a primary low pass characteristic and is periodic in  $n\omega_s$ . As an example, if  $\omega_n$  of the continuous characteristic is  $1/2\pi$  and the sample frequency is 2 kHz, then the minus 3 db point will be 1 Hz. Therefore, a filter shape has been realized. It may be modeled as the upper path in figure 7, with appropriate constants.

At this time other techniques of realization than impulse invariance could be brought to light. These include the Bilinear Z- transformation [5], [8], the Fourier expansion method [5], [11], and the matched Z- transformation [5]. However, the intent is to point out and highlight the underlying theory so instead of clouding the issue with many techniques a practical summary of the theory presented will be given. Further, the limitations of the approach, including quantizing error, error due to roundoff, model selection, and sample rate will be given as a discussion in the design example section.

The above model and approach is most generally useful when the continuous frequency characteristic is known or given. It, as was previously noted, is known as the impulse invariance method [5]. Under this method, which is based upon ideal sampling, it was shown (44) that a pulse transfer function could be derived which represents the unit pulse response of the linear difference equation of the network in the Z or sampled domain. Given a continuous characteristic and a method for calculating its poles and residues this transfer function may be found. The model is then chosen and the filter synthesized. It must be realized, however, that the filter is operating upon time samples of a continuous signal and produces an output which is at the sampling instants. To utilize the filter in a continuous network requires the conversion of these outputs back to a continuous signal. This usually requires hold or smoothing networks.

It is also important to note that each element in the model represents a digital delay network (shift register) or digital hardware for multiplying, scaling, and adding. The size of the bit string that this hardware operates upon will affect the accuracy of the approximation to the continuous characteristic. Even more



important, but not so well known, is that aside from the choice of model affecting hardware complement, the model affects the accuracy of approximation. Also, stability has not been touched upon. If the model and/or other effects causes the poles of the approximation to move outside of the unit circle in the  $Z$  domain, the model will be unstable and diverge from the required approximation [8], [11]. All these points must be considered in the synthesis. They will be discussed in the design example. It may also be noted from the literature, [5], [11], that different methods yield to different models and may even give different results.

In summary, the underlying theory is based upon either ideal sampling and standard  $Z$  transform, or in the other methods upon Fourier expansions and modifications of the standard  $Z$  transform. All yield networks that may be analyzed via linear difference equations or when transformed via  $Z$  transform theory as delay networks. If the theory of sampling and of the model is understood it may be seen (49) that a filter shape is being realized. It is most important to realize that all models are approximations to continuous cases and all include the effects of aliasing or periodicity in their frequency response, due to the sampling or the approximation itself.



### III. N-PATH FILTER DESIGN EXAMPLE

This section is concerned with the presentation of a design example for a N-Path filter. More specifically, it deals with a particular portion of the design area; that of highly selective bandpass filters.

It may be realized from the literature that the high  $Q$  filter problem attracts a great deal of attention. Most of this activity, however, is directed towards the active R-C filter as a means of achieving the desired  $Q$  at the lower frequencies. This approach suffers from high cost, stability problems, and sensitivity. The latter being the hardest and most expensive problem to deal with. While not a cure all for every problem in this area, the structure to be considered in this section offers an attractive alternative to the high selectivity portion of a low frequency filter problem.

In previous sections it was shown that if the path networks have a small bandwidth in comparison to  $N\omega_0$ ,  $(2\pi N/2T)$ , the composite structure follows the zeroth path and exhibits a translated periodic version of  $H(j\omega)$ , centered at the multiples of  $\omega_0$ . To achieve this, bandlimiting had to be employed, either in input/output filtering or in the modulating functions themselves.

In the literature [3], it has been shown that the use of single sinusoidal modulating functions leads to a low pass to band pass transformation on the path network impulse response function  $H(j\omega)$ , without bandlimiting restrictions on input or output of the composite structure. However, in this work we are comparing special purpose networks of Digital Filters to N-Path Filters. In that environment it is most probable that the required sinusoidal modulating functions



are not readily available. It is more likely that pulse functions are available. To this end, we explore pulse modulation, in which the modulating signal is binary in nature.

Consider a single path of the N-Path network, the  $n^{\text{th}}$  path. Let this branch be defined as in figure 8, where  $Z_{ij}$  are the open circuit impedance parameters of the branch.

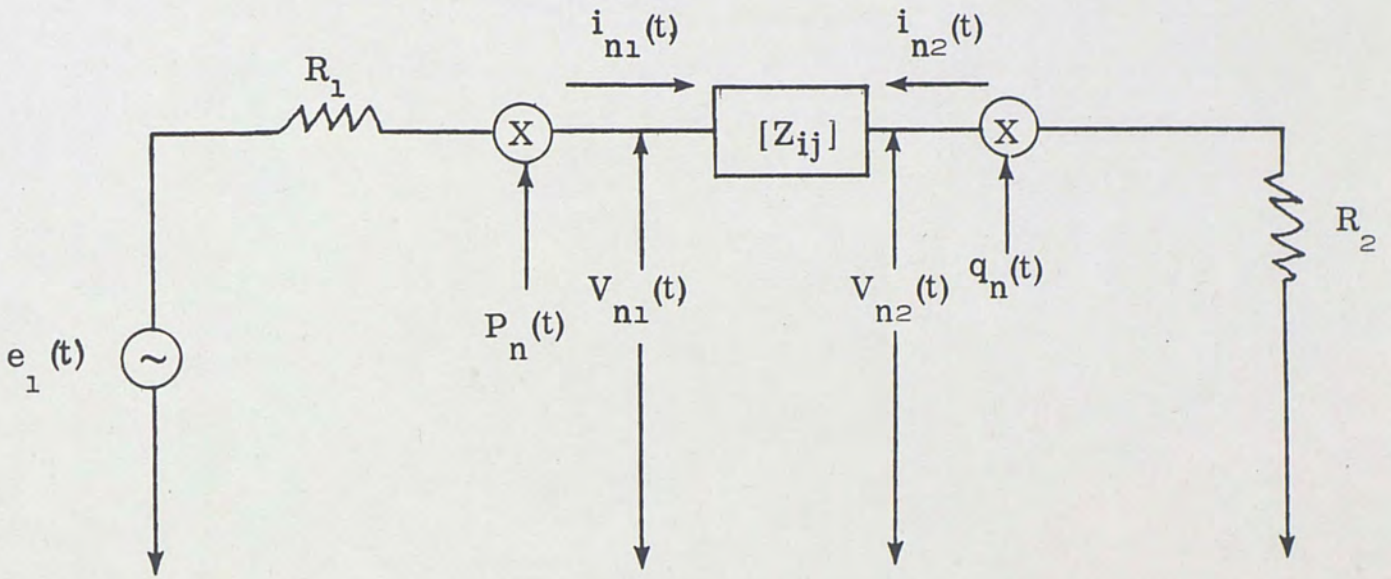


Figure 8.  $n^{\text{th}}$  Path Representation

From this figure then,

$$i_{n1}(t) = \frac{e_1(t) - V_{n1}(t)}{R_1} \cdot P_n(t) \quad (50)$$

$$i_{n2}(t) = \frac{-V_{n2}(t)}{R_2} \cdot q_n(t). \quad (51)$$

If we represent these currents and voltages by their Fourier series and transforms and recall the results of previous sections then

$$I_{n1}(j\omega) = \frac{1}{R_1} \left[ \sum_{m=-\infty}^{\infty} P_m \exp \left[ \left( -j \frac{m2\pi}{T} \right) (n-1) \tau \right] \right] \cdot \left[ E_1(j\omega - jm\omega_0) - V_{n1}(j\omega - jm\omega_0) \right], \quad (52)$$



$$I_{n2}(j\omega) = -\frac{1}{R_2} \left[ \sum_{m=-\infty}^{\infty} Q_m \exp \left[ \left( -j \frac{m2\pi}{T} \right) (n-1)\tau \right] \right] \cdot \left[ V_{n2}(j\omega - jm\omega_0) \right], \quad (53)$$

where  $p_n(t)$  and  $q_n(t)$  are defined as in (18), (19).

Now, the open circuit impedance parameter relationship of the path network may be written as,

$$V_{n1}(j\omega) = Z_{11}(j\omega) I_{n1}(j\omega) + Z_{12}(j\omega) I_{n2}(j\omega), \quad (54)$$

$$V_{n2}(j\omega) = Z_{21}(j\omega) I_{n1}(j\omega) + Z_{22}(j\omega) I_{n2}(j\omega). \quad (55)$$

Substitution of (52), (53), into the above would lead to a set of infinite order difference equations in the voltages. However, since we have assumed that the component networks have a bandwidth small with respect to the  $\omega_0$  of the composite network, we may then make the following simplifications. If

$$|Z_{ij}(j\omega)| \rightarrow 0 \text{ for } |\omega| \geq \frac{\omega_0}{2}, \quad (56)$$

then

$$|V_{n1}(j\omega)| \geq 0 \text{ and} \quad (57)$$

$$|V_{n2}(j\omega)| \geq 0 \text{ for } |\omega| \geq \frac{\omega_0}{2}. \quad (58)$$

Therefore we may eliminate all terms except for  $m = 0$  in the summations involving  $V_{n1}(j\omega)$  and  $V_{n2}(j\omega)$ . Thus for  $|\omega| \leq \omega_0/2$ ,



$$\begin{aligned}
& \left( 1 + \frac{Z_{11}}{R_1} P_o \right) V_{n1} + \frac{Z_{12}}{R_2} Q_o V_{n2} \\
&= \frac{Z_{11}}{R_1} \sum_m P_m \exp \left[ -j \left( \frac{m2\pi}{T} \right) (n-1) \tau \right] E_1(j\omega - jm\omega_o), \quad (59)
\end{aligned}$$

and,

$$\begin{aligned}
& \frac{Z_{21}}{R_1} P_o V_{n1} + \left( 1 + \frac{Z_{22}}{R_2} Q_o \right) V_{n2} \\
&= \frac{Z_{21}}{R_1} \sum_m P_m \exp \left[ -j \left( \frac{m2\pi}{T} \right) (n-1) \tau \right] E_1(j\omega - jm\omega_o). \quad (60)
\end{aligned}$$

Now, if we define  $G(j\omega)$  as

$$G(j\omega) = \frac{Z_{21} R_2 / Q_o}{\left( Z_{11} + \frac{R_1}{P_o} \right) \left( Z_{22} + \frac{R_2}{Q_o} \right) - Z_{12} Z_{22}}, \quad (61)$$

then,

$$V_{n2}(j\omega) = \frac{1}{P_o} G(j\omega) \sum_m P_m \exp \left[ -j \left( \frac{m2\pi}{T} \right) (n-1) \tau \right] E_1(j\omega - jm\omega_o). \quad (62)$$

As in previous sections we multiply this expression by  $Q_n(j\omega)$  and sum over the  $N$  paths. As previously, the resultant exponential forms a geometric series.

We sum that series which yields,

$$V_2(j\omega) = \frac{N}{P_o} \sum_{\ell} Q_{\ell} G(j\omega - j\ell\omega_o) \sum_K P_{KN-\ell} E_1(j\omega - jKN\omega_o). \quad (63)$$



Now, if we assume the same band limiting input/output restrictions, then for  $K = 0$  path,

$$V_2(j\omega) = \frac{N}{P_0} \sum_{\ell} Q_{\ell} P_{-\ell} G(j\omega - j\ell\omega_0) E_1(j\omega). \quad (64)$$

If the structure is followed by a broad band, band pass filter which removes all components for which  $|\ell| \neq 1$  and the  $\ell = 0$  component, then the function becomes,

$$V_2(j\omega) = \frac{N}{P_0} \left[ P_1 Q_{-1} G(j\omega + j\omega_0) + Q_1 P_{-1} G(j\omega - j\omega_0) \right] E_1(j\omega). \quad (65)$$

What must be considered now are the modulating functions  $p(t)$  and  $q(t)$ .

If these are defined as,

$$p(t) = \begin{cases} 1 \\ 0 \end{cases} \quad \text{for } \begin{matrix} 0 \leq t \leq d_1 \\ \text{elsewhere} \end{matrix} \quad (66)$$

and periodic in  $N\tau$ ;

$$q(t) = \begin{cases} 1 \\ 0 \end{cases} \quad \text{for } \begin{matrix} 0 \leq t \leq d_2 \\ \text{elsewhere} \end{matrix} \quad (67)$$

and periodic in  $N\tau$ ,

then,

$$P_{\ell} = \frac{d_1}{N\tau} \exp(-j\ell\pi d_1/T) \text{Sa}\left(\frac{\ell\pi d_1}{T}\right), \quad (68)$$

and

$$Q_{\ell} = \frac{d_2}{N\tau} \exp(-j\ell\pi d_2/T) \text{Sa}\left(\frac{\ell\pi d_2}{T}\right), \quad (69)$$

then we have effected a low pass to band pass transformation, because  $V_2(j\omega)$  becomes, where  $N\tau = T$ ,

$$V_2(j\omega) = \frac{Nd}{T} \left[ a_1 G(j\omega - j\omega_0) + a_1^* G(j\omega + j\omega_0) \right] E_1(j\omega), \quad (70)$$



where,

$$a_1 = \exp \left[ (j\pi/T) (d_1 - d_2) \right] \text{Sa} \left( \frac{\pi d_1}{T} \right) \text{Sa} \left( \frac{\pi d_2}{T} \right). \quad (71)$$

The above demonstrates the low pass to band pass transformation of the component network when periodic binary modulation is used, band limiting input/output filters are employed, the component network bandwidth is small with respect to the modulation frequency, and loading is taken into account. In this realization, the product modulators default to series analog switches that are gated by the modulation waveform. If the structure can be used such that the input is a current source and if the output loading can be made negligible, then (25) may be used directly [3].

Consider (25) again, repeated here,

$$V(j\omega) = N \sum_{\ell} P_{-\ell} Q_{\ell} H \left( j\omega - j \frac{2\pi\ell}{N\tau} \right) U(j\omega). \quad (72)$$

If we again require an output broad band, band pass filter to remove images for  $|\ell| \neq 1$ , and the DC ( $\ell = 0$ ) component, then (72) becomes,

$$V(j\omega) = N \left\{ Q_{-1} P_1 H \left( j\omega + j \frac{2\pi}{N\tau} \right) + P_{-1} Q_1 H \left( j\omega - j \frac{2\pi}{N\tau} \right) \right\} V(j\omega). \quad (73)$$

In this equation  $2\pi/N\tau = \omega_0$ . Further, let the same modulating waveforms be used. The result is again a low pass to band pass transformation

$$V(j\omega) = \frac{N(d_1 d_2)}{T^2} \left\{ a_1 H(j\omega - j\omega_0) + a_1^* H(j\omega + j\omega_0) \right\} V(j\omega), \quad (74)$$

where the  $a_1$  factor is defined as in (71). Thus, two options present themselves in design. If slightly more hardware can be tolerated to effect the current source input, isolated output, one may design directly around a given impulse response.



If the additional hardware cannot be used, then a low pass network must be designed using open circuit impedance parameters and then the transformation effected, including the loading effects.

In all of the above, we made no mention of a choice for  $N$ . To see what  $N$  must be, consider the restrictions of bandlimiting. The whole basis of this approach has been that we bandlimit to the requirement that the bandwidth of the component network be small in comparison with the modulation frequency. Specifically,

$$|\omega_c| \leq \frac{N\omega_o}{2}, \quad (75)$$

so that the network would follow the zeroth path, and such that imaging would occur above  $N\omega_o/2$ . Thus a minimum  $N$  could be chosen at  $N = 3$  to satisfy (75). However, if the dwell times ( $d_i$ ) become small, the product modulators approach impulse samplers. In this case, to preserve frequency spectrum shape  $N$  must be chosen as 4 [6]. We will use the latter as our design baseline. It is well to note however, that by using alternation modulation that  $N$  may be reduced by a factor of 2 [3], [4]. However, such modulators are not easy to achieve. Further, it should be noted that in practice  $\omega_o$  is chosen orders of magnitude above  $\omega_c$ . This is done to ensure that images due to path mismatches fall well outside the region of concern.

Before delving into a design it would be well to review some of the limitations of this structure. The strongest fault lies in the matching of paths. All of the above work assumed identical branches. If this is not the case, the above analyses falls apart. For minor mismatch, the effect is unwanted images. For major mismatch, the effect is unpredictable in general. It would at least cause a time varying network and a non-wanted response. The next major fault lies in input/output filtering (for this approach). Extra hardware is used to make the overall structure appear time invariant. Other faults lie in drift for each path and overall, and in the cost of modulators.



The design example will start with a low pass impulse response. The response will be converted to the frequency domain and then to components. From there an assumed source resistance will be added as well as the assumed load resistance. The equivalent circuit will be drawn from two-port theory and then the choice of  $N$ ,  $\omega_0$  will be made. The predicted response will then be given.

Consider an impulse response of the form,

$$h(t) = \omega_n \exp(-\omega_n t). \quad (76)$$

Then in the frequency domain, this response is given by,

$$H(S) = \frac{\omega_n}{S + \omega_n}. \quad (77)$$

It is to be realized that this is a low pass function. To visualize this, let the complex frequency variable be confined to the imaginary ( $j\omega$ ) axis. Then, in terms of frequency response,

$$H(j\omega) = \frac{1}{j\omega/\omega_n + 1}, \quad (78)$$

and therefore it is seen that the response is down 3 db at  $\omega = \omega_n$  and is low pass in nature. To construct a circuit which exhibits this response consider the network shown in figure 9, and  $E_1(S) = 1$  an impulse at  $t = 0$ .

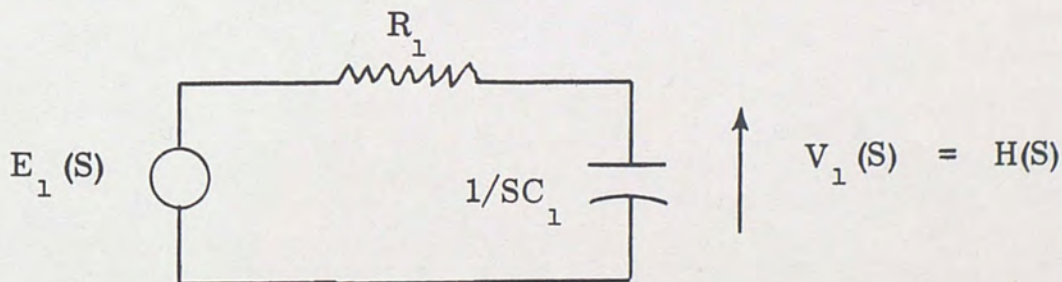


Figure 9. Low Pass RC Network



By voltage division,  $V_1(S)$  is then seen to be,

$$V_1(S) = \frac{1/SC_1}{1/SC_1 + R_1}, \quad (79)$$

which is easily converted to the form,

$$V_1(S) = \frac{1/R_1 C_1}{1/R_1 C_1 + S}. \quad (80)$$

Therefore, for this network,  $\omega_n = 1/R_1 C_1$ . If the second method of realization is used, current source input, negligible loading, and the same binary modulation is used, the response will be down 3 db  $\omega_n$  units away from  $\omega_0$ , for the given bandlimiting restrictions.

Consider now that the first method is to be used. For that approach, we must include source and load resistance. Current integrated circuit analog switches exhibit an "on" resistance of from 50 to 1000 ohms. Assume that the circuit drives a load of 10,000 ohms, a reasonable load when feeding an operational amplifier. Further assume that the source resistance is lumped all within the switches, then,  $R_1 = 1000 \Omega$  and  $R_2 = 10K \Omega$ . We must now define the open circuit impedance parameters of the network. For the circuit of figure 9, these are,

$$\begin{aligned} Z_{11} &= R_1 + 1/SC_1 \\ Z_{12} &= 1/SC_1 \\ Z_{21} &= 1/SC_1 \\ Z_{22} &= 1/SC_1 \end{aligned} \quad (81)$$

Under open circuit output, zero source resistance input, this circuit exhibits the same response as before. Now add the required effects. The circuit appears as in figure 10.



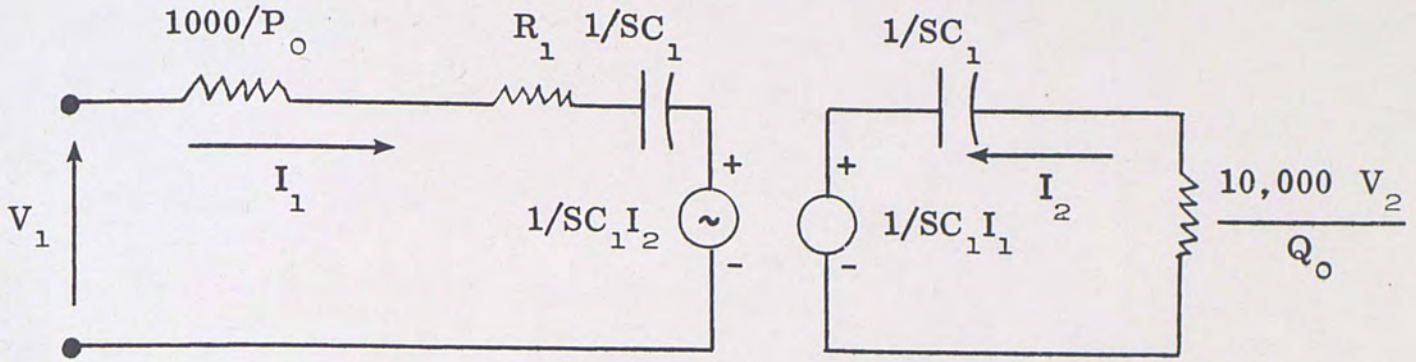


Figure 10. Two Port Representation,  $n^{\text{th}}$  Path

Define  $1000/P_O$  as  $R_S$ , and  $10,000/Q_O$  as  $R_O$ . Then the voltage transfer function of the circuit is given by,

$$\frac{V_2}{V_1} (S) = \frac{(1/SC_1) R_O}{(1/SC_1 + R_O) (R_S + R_1 + 1/SC_1) - 1/S^2 C_1^2} \quad (82)$$

This leads to a gain factor of  $1/C_1 (R_S + R_1)$ , and a minus 3 db point at  $\omega = (R_S + R_1 + R_O)/C_1 (R_1 + R_S) R_O$ . Therefore, both the gain and roll off of the circuit are changed by taking source and load resistance into account. We will show, however, that for small dwell times and non-negligible source/load resistances, this case defaults to the latter case of a current source input, negligible load output.

Consider that  $\omega_o$  is required to be  $2\pi \cdot 10^3$  radians/second. The available analog switches on the market today can respond in 10 microseconds or less. Therefore, let  $d_1 = d_2 = 10$  microseconds.  $R_S$  is then given by,

$$R_S = 1000 \left( \frac{\omega_o}{2\pi d_1} \right), \quad (83)$$

and equals  $10^{11}$  ohms. Similarly,  $R_O$  is given by,

$$R_O = 10\,000 \left( \frac{\omega_o}{2\pi d_2} \right), \quad (84)$$



and equals  $10^{12}$  ohms. We may easily conclude that for most voltage sources  $R_S$  converts them to a current source. Also  $10^{12}$  ohms is quite a negligible load. Therefore, our required  $\omega_c$  can be given by,

$$\omega_c = \omega_n, \quad (85)$$

and is  $1/R_1 C_1$  for the network of figures 9 and 10. Thus, to design the overall structure we will use four paths as given by figure 9.  $\omega_o$  is given as a design requirement as is  $\omega_c$ . Then we choose convenient values for  $R_1$  and  $C_1$ . We may use any simple broad band, band pass filter for the output as long as it removes components at DC and for  $M\omega_o$ , where  $M > 1$  and integer. Similarly, the input filter need only be a low pass section with cut off at  $2\omega_o$ . This being done, the response shape will have arithmetic symmetry about  $\omega_o$  and have a single stage roll off rate (-6 db/octave).

In summary, we have shown that given proper bandlimiting, the N path structure transforms a low pass section into an overall single pass band, band pass filter. It was shown that the minimum number of paths for binary modulation should be 4. Also demonstrated was a method of handling source and load resistance. Most importantly, it was shown that for small dwell times, the model defaults to a much simpler one, involving a single path impulse response. The path network chosen was a simple one. However, any path network may be used to achieve much sharper roll off than in the example. The design approach would remain the same, the price paid being additional resistors and capacitors in each path. The design approach has as its salient points;

1. From the design requirements, design a low pass section to meet  $\omega_c$  from DC.
2. Choose the switch dwell times (binary modulation), small enough to ensure current source input, negligible load output.
3. Use  $N = 4$ , paths.



4. Use a simple low pass input filter at  $\omega_c = 2 \omega_o$ .
5. Use a simple broad band output band pass filter to remove DC and components for frequencies  $M\omega_o$ , where  $M > 1$  and integer.
6. The realized response will have arithmetic symmetry about  $\omega_o$  and follow the path network response shape.
7. If needed compensate for filter attenuation of the signal. At the center frequency the filter gain is given by

$$\text{GAIN} = \left( \frac{4d_2}{T} \right) \text{Sa} \left( \frac{\pi d_1}{T} \right) \text{Sa} \left( \frac{\pi d_2}{T} \right). \quad (86)$$



#### IV. DIGITAL FILTER DESIGN EXAMPLE

The goal of this section is to provide insight into the design of a Digital Filter. Specifically, it is concerned with the presentation of a design example for a high Q band pass filter.

Before any design may be approached, a method must be chosen, and an understanding of the limitations of that procedure should be undertaken. To this end, we will use the method of impulse invariance and the standard Z transform. The errors that arise will be explored first and then the design will be presented.

A band pass filter represented in the complex frequency domain (S plane) would be given as,

$$H(S) = \frac{H_0 \left[ \frac{\omega_2 - \omega_1}{\omega_0} \right] \omega_0 S}{S^2 + \left[ \frac{\omega_2 - \omega_1}{\omega_0} \right] \omega_0 S + \omega_0^2}, \quad (87)$$

where

$\omega_2 \triangleq$  upper cutoff frequency,

$\omega_1 \triangleq$  lower cutoff frequency,

$\omega_0 \triangleq$  center frequency, and

$H_0 \triangleq$  gain at center frequency.

Usually the poles of this function are complex conjugate in nature. Defining these roots as follows, leads to the form that is needed in the sampled domain



(Z plane). Let the form (87) be put in partial fraction form,

$$H(S) = \frac{R_r + jR_i}{S + \alpha - j\beta} + \frac{R_r - jR_i}{S + \alpha + j\beta} \quad (88)$$

Then using the method of complex convolution, and closing the contour to the left about the singularities of  $H(S)$  yields,

$$H(Z) = \frac{A_1 Z^{-1} + A_0}{B_2 Z^{-2} + B_1 Z^{-1} + 1}, \quad (89)$$

where the coefficients are defined as,

$$\begin{aligned} A_1 &\triangleq -2T \exp(-\alpha T) R_r \cos(\beta T) + R_i \sin(\beta T), \\ A_0 &\triangleq 2TR_r, \\ B_2 &\triangleq \exp(-2\alpha T), \text{ and} \\ B_1 &\triangleq -2 \exp(-\alpha T) \cos(\beta T). \end{aligned} \quad (90)$$

To approach the design requires a network realization of (89). In deriving this realization, the following questions must be answered:

1. Input quantizer—what is sample rate and step size?
2. What is the required bit string of the model (word size)?
3. What form of model should be chosen?
4. What form of arithmetic should the model use?
5. What choices of the above, for the particular problem, lead to a stable model.

The answers to these questions dictate the approach to be followed. Hence, they will be discussed here.



In previous sections it was shown that in order to avoid aliasing in the frequency response, the sample rate should be twice the highest pole frequency or twice the highest frequency present in the incoming data. These may differ and can cause problems. The reason is that the sample rate, the word length and the pole frequency are inter-related. Consider the high Q problem. Then  $\alpha$  in equation (88) will be very small. In equation (90) it may be seen that for a small  $\alpha T$ ,  $B_2$  becomes [5],

$$B_2 \approx -(1 - \alpha T). \quad (91)$$

Consider now the sign magnitude binary representation. If only N bits are available for representing  $B_2$ , then the smallest  $\alpha T$  that may be represented is just  $2^{-N}$ . Therefore, the largest  $B_2$  is,

$$B_{2_{\max}} = -(1 - 2^{-N}). \quad (92)$$

This implies that one can find N as,

$$N = \frac{-\log(1 + B_2)}{\log 2} = \frac{-\log \alpha T}{\log 2} \quad (93)$$

Thus, if T is fixed, since  $\alpha$  is a given quantity, N is determined. Conversely, if N is fixed, T is then known. But which T to use. If the pole frequency requires a sample rate  $T_1$ , and N is free, then there is no problem because for a given minimum  $\alpha$ , N may be adjusted. Similarly, if the incoming data requires a different sample rate,  $T_2$ , and N is again free, it may be adjusted to give the minimum required  $\alpha$ . An example should clarify this. Assume that the S plane pole Q is 4000. Q is defined as,

$$Q \triangleq \frac{\beta}{\alpha}. \quad (94)$$



One would then wish to sample at  $2\beta$  from the considerations of previous sections. Therefore,  $\beta$  would equal  $1/2T$ , and  $\alpha T$  would equal  $1/2 Q$ . This implies an  $\alpha T$  of  $.125 \cdot 10^{-3}$ . This means  $N$  would have to be 14. If the sample rate were higher and the number of bits fixed at 14, the particular value of  $\alpha$  for this example could not be realized.

The above implies that the sample rate and word size are to be derived from the required filter pole components. This in turn fixes the quantizer step size.

When the word size has been determined, then each coefficient will be represented by a finite word length. Because of this, the pole locations in the  $Z$  plane are not the exact pole locations. What must be done, is to examine the locations under the required word size. For complex poles, this results in four possible locations for each of the required poles [5]. These locations are,

$$\begin{aligned}
 P_1 &= \frac{-[B_1]}{2} + j \left[ [B_2] - \left( \frac{[B_1]}{2} \right)^2 \right]^{1/2}, \\
 P_2 &= -\left( \frac{[B_1] + \Delta}{2} \right) + j \left[ [B_2] - \left( \frac{[B_1] + \Delta}{2} \right)^2 \right]^{1/2}, \\
 P_3 &= \frac{-[B_1]}{2} + j \left[ [B_2] + \Delta - \left( \frac{[B_1]}{2} \right)^2 \right]^{1/2}, \text{ and} \\
 P_4 &= -\left( \frac{[B_1] + \Delta}{2} \right) + j \left[ [B_2] + \Delta - \left( \frac{[B_1] + \Delta}{2} \right)^2 \right]^{1/2},
 \end{aligned} \tag{95}$$

where quantities in  $[ ]$  signify the truncated value for  $N$  bits and  $\Delta$  represents the step size of  $2^{-N}$ . The location chosen would logically be the one closest to the required location. However, if this lies on the unit circle, one would best choose a different location because a pole on or exterior to the unit circle leads to instability, since the mapping is the right half  $S$  plane.



The above discussion implies that whatever model is chosen, the quantized pole locations should be examined, and positions used which are interior to the unit circle.

The choice of model for this type of filter realization is not one which presents many choices. In larger digital systems, where much more complex  $S$  plane models are to be quantized, the choice is between the models known as direct, canonical, cascade and parallel forms. For this problem, we are realizing only one filter section, so that the choice is based upon minimizing hardware, improving accuracy, and achieving stability. Consider then how this filter function (89) may be realized. At least two realizations are possible. These are shown in figures 11 and 12.

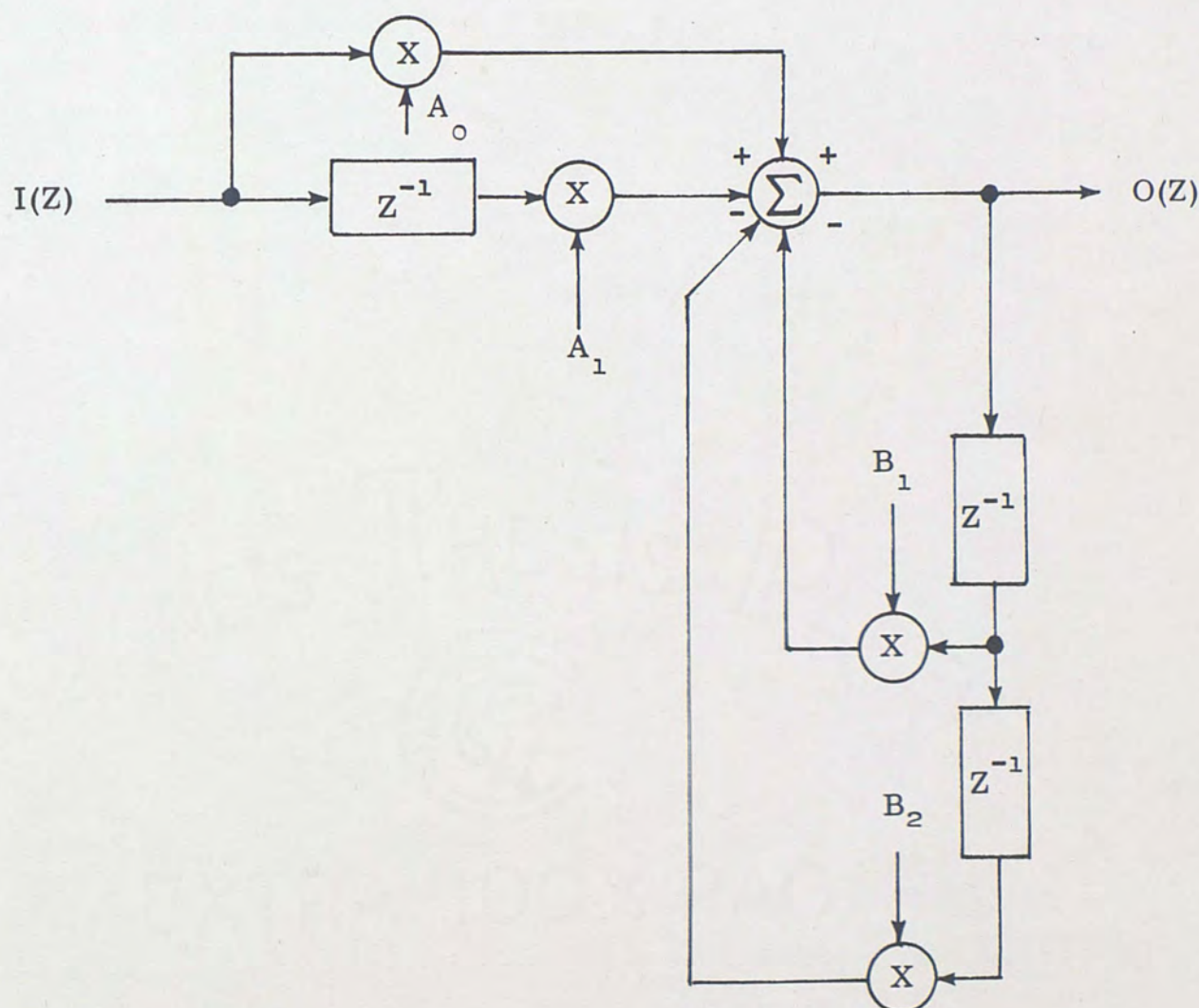


Figure 11. Second Order Function, Model 1



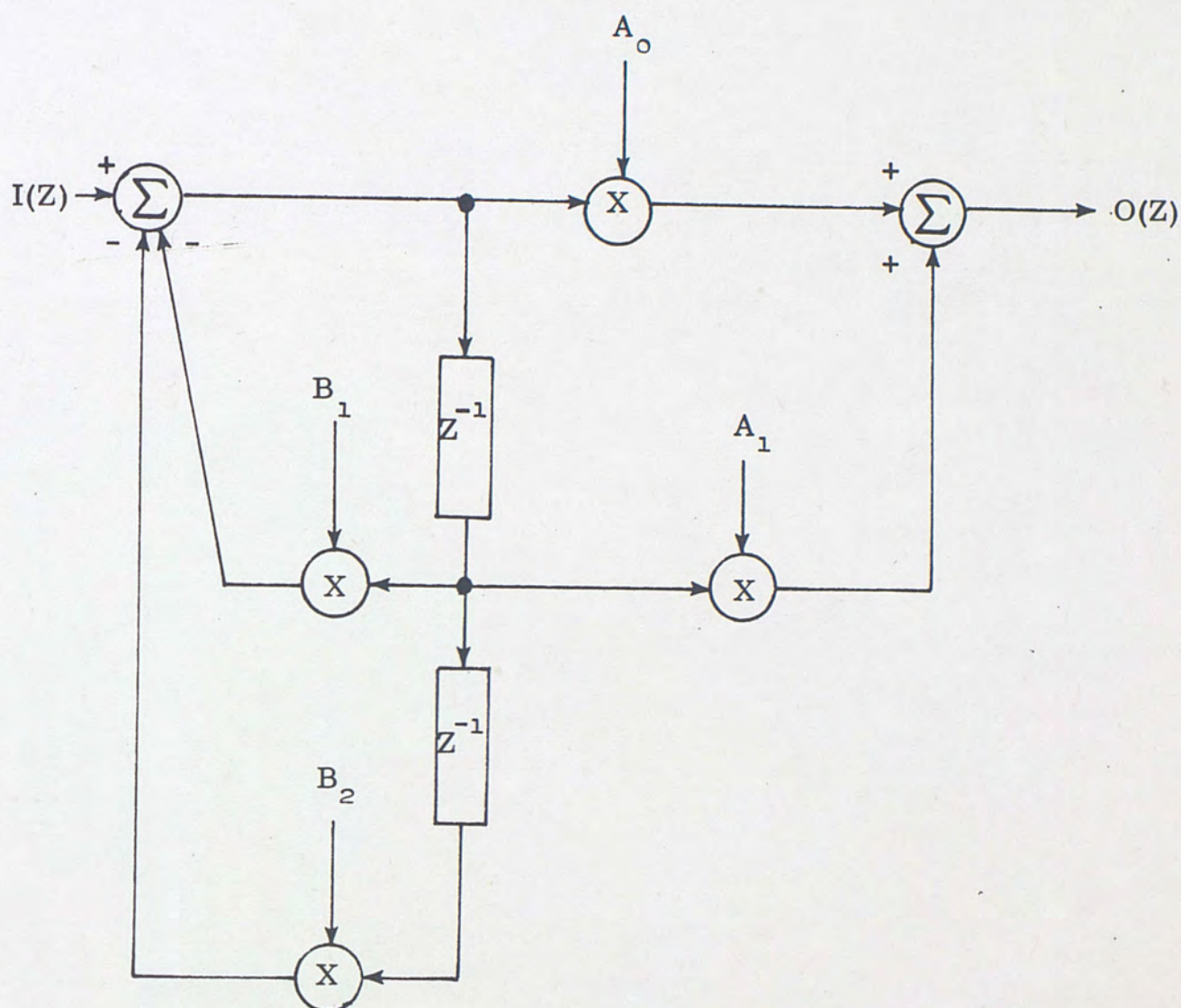


Figure 12. Second Order Function, Model 2

From these figures it must be ascertained which is the most economic in hardware. It is seen that in either case four multiplier circuits are required. Further, the first figure trades adder circuitry for a register. It is to be concluded that the network of figure 11 is the one to use, because in general, register circuits are less costly than adder circuits.

Within the chosen model, the question of which form of arithmetic to use must also be answered. To approach an answer the two forms of arithmetic must be considered. These are called fixed point and floating point notation.



In fixed point notation, all variables are scaled and the binary point is fixed. In floating point notation, the number is represented in two parts, the exponent and the mantissa. In building hardware, especially multipliers, it has been found that the scaled variable, fixed point notation is more economical. It must be realized that to achieve the required accuracy in fixed point multipliers requires double precision adders with bit chopping at the output. The design baseline will then be the use of scaled variable, fixed point arithmetic with double precision adders internal to the multiplier networks.

From the above discussion, the following will be the design baseline for the high Q Digital Filter:

1. Determine the required word length and sample rate from the S plane filter characteristic pole  $Q$ , using the relations (93) and (94).
2. Use the now known word size to determine the possible pole locations in the Z plane. Choose values closest to required poles and interior to the unit circle.
3. Use the model of figure 11.
4. Use scaled variable, fixed point arithmetic, with the above multiplier cautions.

Another area that requires a look at error contribution is the input quantizer itself. The input circuit must convert an analog input sample to a digital representation. The input circuit has a finite conversion time which places an upper limit upon the sample rate. Further, it introduces errors due to imperfect sampling and due to the converter accuracy itself. The best method of attack is to use the error parameters supplied by the converter vendor to generate a worst case input error signal. Use the transfer function of the ideal filter operating upon the input error to determine an output error bound. This is possible because of the assumed linear nature of the filter. In practice, the filter-converter combination will usually never approach this bound.



We are now in a position to approach the high Q Digital Filter design. We start with a given requirement or set of requirements. These are usually the center frequency, the upper and lower cutoff frequencies, and the center frequency gain. With these parameters given, in the S plane one would locate the required poles. Notice that for a band pass filter, there is always a transmission zero at the origin. We will use the same center frequency as was used in the N-Path Filter design example,  $2\pi \cdot 10^3$  radians/second. Further, we will specify that the bandwidth required is 10 radians/second and that the center frequency gain be unity. This leads to a filter function in the S plane of,

$$H(S) = \frac{(2\pi)(10)S}{S^2 + (2\pi)(10)S + (2\pi \cdot 10^3)^2} \quad (96)$$

The poles of this function lie at,

$$S_{1,2} = -31.42 \pm j6283.11, \quad (97)$$

and a pole Q of 199.97. This yields the following function in the Z plane, where the sample rate is twice  $\beta$ , or T is  $5 \cdot 10^{-4}$ ;

$$H(Z) = \frac{.0309Z^{-1} + .0314}{.9691Z^{-2} + 1.96882Z^{-1} + 1} \quad (98)$$

We choose N from (93) to be 6. Therefore, the following parameters are available for our filter design:

$$N = 6, \text{ and}$$

$$T = 5 \cdot 10^{-4}.$$

This gives actual pole locations in the Z plane as,

$$Z_{1,2} = -.98441 \pm j.00612. \quad (99)$$



The truncated values force one of the poles outside of the unit circle. Therefore, using equation (95) we find  $P_3$  closest to the required poles and within the unit circle. We therefore modify (98) and get a new  $H(Z)$  as equation (101), where the poles are now,

$$Z_{1,2} = -.94688 \pm j.01732. \quad (100)$$

$$H(Z) = \frac{.0309Z^{-1} + .0314}{.89687Z^{-2} + 1.89375Z^{-1} + 1}. \quad (101)$$

We now may implement this filter in hardware using the model of figure 11.

The design example may now be summarized. We started with a known S plane filter function. Given the design requirements we obtain the poles of that function and convert to the Z plane. Using the required number of bits, we test the Z plane poles for stability and choose quantized pole locations closest to the required poles. We then implement in hardware per figure 11. This is the procedure that is followed under impulse invariance and the standard Z transform. For this class of problem it yields satisfactory results.

The important points to note are:

1. The sample rate is determined by pole frequency.
2. The word size is fixed once the sample rate and poles are known.
3. Truncation must be accounted for to ensure stability and  $H(Z)$  so modified.
4. The network of figure 11 is in general the most economical for this class of problem.



## V. DESIGN COMPARISON

In this section we will compare the N-Path and Digital Filter designs in terms of complexity, parts cost and ease of fabrication. This is necessary if we will be able to choose tractable design alternatives, and use the most economical design for the particular task.

### Design Complexity

The N-Path Filter design in terms of the current source input, negligible load output is not a very complex design. Once the given requirements are known, minus 3 db point and roll off shape, the design may proceed in terms of elemental low pass sections. The theory of multipole low pass responses is well developed. Once the low pass section(s) are designed, the rest of the process entails the choice of analog switches and the design of simple filters for the input and output. Binary modulation is assumed, and the last design step involves a multiphase clock signal to be used for the binary modulation source.

The Digital Filter design is also not very complex in terms of the given requirements. The complex frequency domain (S plane) response shape is fixed. Under impulse invariance the Z plane response is also fixed. Given the requirements for center frequency, bandwidth, and center frequency gain, the design proceeds with the determination of the poles of the given S plane function. This function is then put in partial fraction form and the coefficients of the Z form are evaluated once the sample rate is determined. The number of bits in the word is then selected and the truncated pole values are checked in the Z plane to ensure stability. Once the final Z form coefficients are found, the design proceeds with commitment to hardware. The only other choices required are the



input analog to digital converter and the output digital to analog converter. The word size and sample rate chosen above will dictate the input and output converters.

In comparing the complexity of the two designs, it is seen that the basic complexity, or lack thereof, in the N-Path Filter is in the path networks. Designing these to meet shape specifications poses the design problem. In the Digital Filter, the shape and cutoff are predefined by the given response used in the impulse invariance transformation from S to Z planes. We chose a simple shape and rolloff rate. If others are required, the design must be reformulated in the S plane. However, one can use low pass prototypes and then the low pass to band pass transformation to achieve the desired shape. In that sense, the design of both types of filters is quite similar in complexity.

### Parts Cost

The N-Path Filter that we designed was composed of four identical paths. Each path contained an input series switch, a passive path R-C network, and an output series switch. The input to the structure was an active or passive low pass filter, and the output of the structure was a broad band, band pass filter, using operational amplifier techniques. In terms of integrated circuit packages, the input filter may be constructed in three packages for an active network as can the output filter. The series switches require only two packages. Depending upon the complexity of the path networks, these may require only a single package or as many discrete carriers as it takes to build multipole responses. The modulation functions require six packages including the clock oscillator. Thus from as little as fifteen packages the overall structure may be constructed. In terms of power supplies, both analog and digital supplies will be needed. These are usually in the system anyway, but it is well to point out the need for the analog supply.



The Digital Filter that was designed consisted of an input converter (A/D), three registers, four multipliers, one adder, and an output converter (D/A). In terms of package count we require a single package each for the input and output converters, twelve packages for each multiplier (based on parallel arithmetic and current technology, including algorithm), and nine packages to implement the multistage adder networks. A clock circuit is also required at five packages. Thus to build this filter requires sixty-seven packages including three for the registers. This structure also requires analog and digital power supplies; the analog ones for the converters.

In terms of parts count, the N-Path filter is about four times more economical. It is true, however, that for more complex shapes of response, the Digital Filter may or may not change in parts count, the N-Path Filter must increase. Experience has shown that for most shapes, the N-Path Filter is approximately twice as economical in parts than the corresponding Digital Filter. Thus, in comparison, for similar tasks and for the class of problem under consideration, the N-Path Filter is to be preferred from an economics point of view.

### Fabrication

In packaging and building both types of designs, fabrication would be no problem. The N-Path Filter, if it uses discretes, will require more manual operations than the all integrated circuit Digital Filter. Because of the higher package count, the Digital Filter requires more wiring. In general, the fabrication cost for both approaches, excluding parts, is similar.

In summary, the designs when compared in terms of complexity, parts and fabrication, for the high  $Q$ , band pass filter problem under consideration, shows that the N-Path Filter structure, as designed herein, is the more economical of the two approaches.



## VI. SUMMARY AND CONCLUSIONS

In this work we have explored the underlying theory of N-Path and Digital Filters. It has been found for the N-Path Filter that:

- Basic structure is a time varying network, which under certain band-limiting conditions can be made to behave as a time invariant filter network.
- Using the bandlimiting conditions, a single pass band, band pass filter can be derived.
- The characteristics of the band pass filter are dependent upon the path network and the modulation function, however, the response must follow all rules of physical realizability.
- The form of response is of a low pass to band pass transformation, which is a translation of the path response, with arithmetic symmetry about DC, to one with arithmetic symmetry about  $\omega_0$ , the switching frequency.
- Depending upon the complexity of the path network, very stringent characteristics about  $\omega_0$  may be met.

For the Digital Filter, it was learned that:

- Filter response is an approximation to the required characteristic in the complex frequency (S plane) domain.
- Sample rate must be at least twice the highest pole frequency in the S plane to avoid aliasing in the Digital Filter frequency response.
- Sample rate, pole frequency and word size in the approximation are inter-related.



- Response shape is directly related to the characteristic chosen in the S plane.
- Pole migration due to truncation (word size) must be accounted for to avoid stability problems.
- The model chosen affects the accuracy of the approximation.
- Since the response is related only to the S plane function, any function which can be modeled in the S plane can be approximated; only those with poles interior to Z plane unit circle will be stable.
- Methods other than impulse invariance and the standard Z transform may be employed. In general, these lead to different realizations and results.

This effort has been concerned with comparison of N-Path and Digital Filters for high Q band pass problems. It has been shown that for this particular class of problem, and for equal response shape, the N-Path Filter yields equal performance at much reduced parts cost. This result is achieved at the following expense:

1. All paths must be identical.
2. Parts count (cost) goes up directly with shape stringency.
3. Imaging or unwanted carrier frequency leakage is a function of path match and the goodness of the band limiting techniques employed.

The Digital Filter we are comparing against is parts expensive, even for non-stringent shape requirements, as we have demonstrated. If the requirements are tightened, in general, additional filter sections must be employed. In the N-Path Filter, this is handled by adding additional pole producing sections to each path network.

It must be observed, that for most common requirements the N-Path Filter serves the need, and in a more economical fashion, as demonstrated. It is the



conclusion of this work, therefore, that N-Path Filters serve as a viable alternative to Digital Filters for the problem under discussion, and should be considered for use in designs approaching this area of activity. The key point is economy of design, and this may be achieved as long as the path networks required are not so unwieldy as to make the overall design ineffectual when compared with a Digital Filter to do the same work on a parts cost basis.



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